pie hat starred alongside Lippy the Lion in dozens of animations.

So, the punch line involves a pun based on $r \, dr \, r$, but why is this the answer to the mathematics question? The teacher has posed a problem that relates to a notoriously nasty area of mathematics known as calculus. This is a topic that strikes terror into the hearts of many teenagers and triggers nightmarish flashbacks in some older people. As the teacher explains when she sets out the problem, the goal of calculus is to “determine the rate of change” of one quantity, in this case $y$, with respect to another quantity, $r$.

If you have some recollection of the rules of calculus, then you will be able to follow the logic of the joke fairly easily and arrive at the

* Readers with a rusty knowledge of calculus may need to be reminded of the following general rule: The derivative of $y = r^n$ is $dy/dr = n \times r^{n-1}$. Readers with no knowledge of calculus can be reassured that their blind spot will not hinder their understanding of the rest of the chapter.
ready estimate of 3.14, and to pin down this elusive number by measuring it as accurately as possible.

The first serious attempt at a more precise measurement of \( \pi \) was made by Archimedes in the third century B.C. He could see that an accurate measurement of \( \pi \) depended on an accurate measurement of the circumference of a circle. This is inevitably tricky, because circles are built from sweeping curves, not straight lines. Archimedes’s great breakthrough was to sidestep the problem of measuring curves by approximating the shape of a circle with straight lines.

Consider a circle with diameter \((d)\) equal to 1 unit. We know that \( C = \pi d \), which means it has a circumference \((C)\) equal to \( \pi \). Next draw two squares, one around the outside of the circle and one tucked inside the circle.

![Diagram of a circle and two squares](image)

The actual circumference of the circle must, of course, be smaller than the perimeter of the large square and larger than the perimeter of the small square. So, if we measure the perimeters of the squares, we can obtain upper and lower bounds on the circumference.

The perimeter of the large square is easy to measure, because each side is the same length as the circle’s diameter, which we know to be equal to 1 unit. Therefore the perimeter of the large square measures \( 4 \times 1 = 4 \) units.

The perimeter of the small square is a little trickier to fathom, but we can pin down the length of each side by using the Pythagorean theorem. Conveniently, the diagonal of the square and two of its sides...
form a right-angled triangle. The hypotenuse \((H)\) is not only equal in length to the diagonal of the square, but it is also as long as the circle’s diameter, namely 1 unit. The Pythagorean theorem states that the square of the hypotenuse is equal to the sum of the squares of the other two sides. If we label the sides of the square \(S\), then this means that \(H^2 = S^2 + S^2\). If \(H = 1\), then the other two sides must each have a length of \(1/\sqrt{2}\) units. Therefore the perimeter of the small square measures \(4 \times 1/\sqrt{2}\) units = 2.83 units.

As the circumference of the circle must be less than the perimeter of the large square, yet greater than the perimeter of the small square, we can now declare with confidence that the circumference must be between 2.83 and 4.00.

Remember, we stated earlier that a circle with a diameter of 1 unit has a circumference equal to \(\pi\), therefore the value of \(\pi\) must lie between 2.83 and 4.00.

This was Archimedes’s great discovery.

You might not be impressed, because we already know that \(\pi\) is roughly 3.14, so a lower bound of 2.83 and an upper bound of 4.00 are not very useful. However, the power of Archimedes’s breakthrough was that it could be refined. For, instead of trapping the circle between a small and a large square, he then trapped the circle between a small and a large hexagon. If you have ten minutes to spare and some confidence with numbers, then you can prove for yourself that measuring the perimeters of the two hexagons implies that \(\pi\) must be more than 3.00 and less than 3.464.
of Homer” (1993). Newton is one of the fathers of modern mathematics, but he was also a part-time inventor. Some have credited him with installing the first rudimentary flapless cat flap—a hole in the base of his door to allow his cat to wander in and out at will. Bizarrely, there was a second smaller hole made for kittens! Could Newton really have been so eccentric and absentminded? There is debate about the veracity of this story, but according to an account by J. M. F. Wright in 1827: “Whether this account be true or false, indisputably true is it that there are in the door to this day two plugged holes of the proper dimensions for the respective egresses of cat and kitten.”

The bits of mathematical scribbling on Homer’s blackboard in “The Wizard of Evergreen Terrace” were introduced into the script by David S. Cohen, who was part of a new generation of mathematically minded writers who joined The Simpsons in the mid-1990s. Like Al Jean and Mike Reiss, Cohen had exhibited a genuine talent for mathematics at a young age. At home, he regularly read his father’s copy of Scientific American and toyed with the mathematical puzzles in Martin Gardner’s monthly column. Moreover, at Dwight Morrow High School in Englewood, New Jersey, he was co-captain of the mathematics team that became state champions in 1984.
is to start by taking Maggie across the river from the original bank to the destination bank. Then he would return to the original bank to collect the poison, and row back to the destination bank and deposit the poison. He cannot leave the poison with Maggie, so he would bring Maggie back to the original bank and leave her there, while he takes Santa’s Little Helper across to the destination bank to join the poison. He would then row back to the original bank to collect Maggie. Finally, he would row to the destination bank to complete the challenge with everyone and everything having safely crossed the river.

Unfortunately, he is unable to fully implement his plan. For when Homer leaves Maggie on the destination bank, at the end of the first stage, she is promptly kidnapped by nuns. This is something that Alcuin failed to factor into his original framework for the problem.

In an earlier episode, “Lisa the Simpson” (1998), a puzzle plays an even more important role by triggering the entire plotline. The story starts in the school cafeteria, where Lisa sits opposite Martin Prince, who is perhaps Springfield’s most gifted young mathematician. Indeed, Martin experiences life from an entirely mathematical perspective, as demonstrated in “Bart Gets an F” (1990), in which Bart temporarily befriends Martin and offers him some advice: “From now on, you sit in the back row. And that’s not just on the bus. It goes for school and church, too . . . So no one can see what you’re doing.”

Martin then reframes Bart’s advice in terms of mathematics: “The potential for mischief varies inversely to one’s proximity to the authority figure!” He even jots down the equation that encapsulates Bart’s wisdom, in which $M$ represents the potential for mischief and $P_A$ is proximity to an authority figure:
In the cafeteria, Martin becomes interested in Lisa’s lunch, which is not the usual cafeteria food, but rather a vacuum-packed space-themed meal. When Lisa holds up the lunch and explains that it is “what John Glenn eats when he’s not in space,” Martin spots a puzzle on the back of the packet. The challenge is to find the next symbol in this sequence:

```
Moby Dick
```

Martin solves the puzzle in the blink of an eye, but Lisa remains perplexed. She gradually becomes more and more frustrated as students sitting nearby, including Bart, say that they can identify the next symbol in the sequence. It seems that everyone can work out the answer . . . except Lisa. Consequently, she spends the rest of the episode questioning her intellectual ability and academic destiny. Fortunately, you will not have to suffer such emotional turmoil. I suggest you spend a minute thinking about the puzzle, and then take a look at the answer provided in the caption on the next page.

The lunch puzzle is noteworthy because it helped to shore up the mathematical foundations of *The Simpsons* by playing a part in attracting a new mathematician to the writing team. J. Stewart Burns had studied mathematics at Harvard before embarking on a PhD at the University of California, Berkeley. His doctoral thesis would have involved algebraic number theory or topology, but he abandoned his research before making much progress, and he received a master’s degree instead of a PhD. The reason for his premature departure from Berkeley was a job offer from the producers of the sitcom *Unhappily Ever After*. Burns had always harbored ambitions to become a television comedy writer, and this was his big break. Soon he became friends with David S. Cohen, who invited Burns to the offices of *The Simpsons* in order to attend a table reading of an episode, which happened to be “Lisa the Simpson.” As the storyline unfolded, including the number-based puzzle, Burns gradually felt that this was where he belonged, working alongside Cohen and the other mathematical writers. While working on *Unhappily Ever After*, Burns was labeled as the
Although David S. Cohen cannot remember if he suggested the puzzle that appears in “Lisa the Simpson,” he certainly drew the initial sketches. The puzzle, almost as it appeared in the episode, is in the lower line of this page of doodles. Solving the problem relies on noticing that the left and right halves of each symbol are mirror images of each other. The right half of the first symbol is 1, and the left half is its reflection. The right half of the second symbol is 2, and the left half is its reflection. The pattern continues with 3, 4, and 5, so the sixth symbol would be 6 joined to its own reflection.

The upper line suggests Cohen was thinking of using the sequence (3, 6, 9), but this idea was abandoned, probably because the fourth element, 12, would have required two digits. The middle line, which shows the sequence (1, 4, 2, 7), was also abandoned. It is unclear what the fifth element of the sequence would have been, and Cohen can no longer remember what he had in mind.

deedy mathematician with a master’s degree. By contrast, when he joined The Simpsons, a master’s degree in mathematics was no longer exceptional. Instead of being labeled a geek, he became known as the go-to guy for toilet humor.

After telling me the story about how he was recruited to join The Simpsons, Burns drew some parallels between puzzles and jokes, and suggested that they have a great deal in common. Both have carefully constructed setups, both rely on a surprise twist, and both effectively
By contrast, there are many jokes in which the humor relies on the actual language and tools of mathematics. For example, there is one well-known joke that was apparently created during an exam by a mischievous student named Peter White from Norwich, England. The question asked students for an expansion of the bracket \((a + b)^n\). If you have not come across this type of question previously, then all you need to know is that it concerns the binomial theorem and the correct answer ought to have explained that the \(r\)th term of the expansion has the coefficient \(n! / [(r - 1)! (n - r + 1)!]\). This is quite a technical answer, but Peter had a radically different interpretation of the question and an inspired solution:

Peter’s imaginative answer got me thinking. Creating a mathematical joke requires an understanding of mathematics, and appreciating the joke requires a similar level of understanding. Hence, mathematical jokes test your mathematical knowledge.

With this in mind, I have gathered the world’s best mathematical jokes, classified them according to their degree of difficulty, and divided them into five examination papers distributed through the course of this book. As you continue exploring the mathematical hu-
Like Palmer and James, Richard Cramer was another part-time amateur statistician who would use mathematics to explore baseball. As a researcher with the pharmaceutical company SmithKline, Cramer had access to considerable computing power, which was supposed to be used to help develop new drugs. Instead, Cramer left the computers running overnight in order to tackle questions in baseball, such as whether or not clutch hitters are a real phenomenon. A clutch hitter is a player who has the special ability of excelling when his team is under the most pressure. Typically, the clutch hitter delivers a big hit when his team is on the verge of losing, particularly in a big game situation. Commentators and pundits have sworn for decades that such players exist, but Cramer decided to check: Do clutch hitters really exist, or are they merely the result of selective recall?

Cramer’s approach was simple, elegant, and entirely mathematical. He would measure players’ performances in ordinary games and in high-pressure situations during a particular season—Cramer chose
The three interesting numbers, as they appeared on the Jumbo-Vision screen, would have seemed arbitrary and innocuous to casual viewers, but those with mathematical minds would immediately have seen that each one is remarkable in its own way.

The first number, 8,191, is a prime number. Indeed, it belongs to a special class of prime numbers known as Mersenne primes. These are
A prime-time show

five-option version called rock-paper-scissors-lizard-Spock (RPSLSp). Invented by computer programmer Sam Kass, this version became famous after it was featured in “The Lizard-Spock Expansion” (2008), an episode of the nerd-friendly sitcom *The Big Bang Theory*. Here are the circular hierarchy and hand gestures for rock-paper-scissors-lizard-Spock.

As the number of options increases, the chance of a tie decreases as $\frac{1}{N}$. Therefore, the chance of a tie is $\frac{1}{3}$ in RPS and $\frac{1}{5}$ in RPSLSp. If one wants to minimize the risk of a tie, then the biggest and best available version of RPS is RPS-101. Created by the animator David Lovelace, it has 101 defined hand gestures and 5,050 outcomes that result in a clear win. For example, quicksand swallows vulture, vulture eats princess, princess subdues dragon, dragon torches robot, and so on. The chance of a tie is $\frac{1}{101}$, which is less than 1 percent.

The most intriguing piece of mathematics that has emerged from
net worth of approximately $50 billion. Buffett’s picture in the 1947 Woodrow Wilson High School senior yearbook has the astute caption “Likes math; future stockbroker.”

Buffett is known to be a fan of nontransitive phenomena and sometimes challenges people to a game of dice. Without giving any explanation, he hands his opponent three nontransitive dice and asks him or her to choose first. The opponent feels that this confers an advantage, because this appears to be an opportunity to select the “best” die. Of course, there is no best die, and Buffett deliberately chooses second to allow himself the privilege of selecting the particular die that is stronger than whichever one was chosen by his opponent. Buffett is not guaranteed to win, but the odds are heavily stacked in his favor.

When Buffett tried this trick on Bill Gates, the founder of Microsoft was immediately suspicious. He spent a while examining the dice and then politely suggested that Buffett should choose his die first.
trickier, because there are six possible starting arrangements. Depending on the starting arrangement, the number of flips required to reach the correct arrangement varies from zero to a worst-case scenario of three, so $P_3 = 3$.

In most cases, you can work out for yourself how to obtain the correct order in the appropriate number of flips. However, for the worst-case scenario, the reordering process is not obvious, so this series of three flips is shown below. Each row indicates the action of one flip, namely where the spatula is inserted and the pancake order after the flip.

As the pile of pancakes grows, the problem becomes increasingly difficult as there are more and more possible starting arrangements,
and an increasing number of possible flipping procedures. Worse still, there seems to be no pattern in the series of pancake numbers ($P_n$). Here are the first nineteen pancake numbers:

<table>
<thead>
<tr>
<th>$n$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>$P_n$</td>
<td>0</td>
<td>1</td>
<td>3</td>
<td>4</td>
<td>5</td>
<td>6</td>
<td>8</td>
<td>9</td>
<td>10</td>
<td>11</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$n$</th>
<th>11</th>
<th>12</th>
<th>13</th>
<th>14</th>
<th>15</th>
<th>16</th>
<th>17</th>
<th>18</th>
<th>19</th>
<th>20</th>
</tr>
</thead>
<tbody>
<tr>
<td>$P_n$</td>
<td>13</td>
<td>14</td>
<td>15</td>
<td>16</td>
<td>17</td>
<td>18</td>
<td>19</td>
<td>20</td>
<td>22</td>
<td>?</td>
</tr>
</tbody>
</table>

The sheer difficulty in running through all the pancake permutations and possible flipping strategies means that even very powerful computers have not yet been able to calculate the twentieth pancake number. And, after more than three decades, nobody has been able to sidestep the brute force computational approach by finding a clever equation that predicts pancake numbers. So far, the only breakthroughs have been in finding equations that set limits on the pancake number. In 1979, the upper limit for the pancake number was shown to be less than $(5n + 5)/3$ flips. This means that we can take a foolishly large number of pancakes, such as a thousand, and know for a fact that the pancake number (i.e., the number of flips required to rearrange the pancakes into size order in the worst-case scenario) will be less than

$$\frac{(5 \times 1,000 + 5)}{3} = 1,668\frac{1}{3}$$

Thus, given that you cannot perform a third of a flip, $P_{1,000}$ is less than or equal to 1,668. This result is famous, because it was published in a paper that was co-authored by William H. Gates and Christos H. Papadimitriou. William H. Gates is better known as Bill Gates, co-founder of Microsoft, and this is thought to be the only research paper that he has ever published.

The Gates paper, based on work he did as an undergraduate at Harvard, also mentions a devious variation of the problem. The *burnt*
plant did contain nicotine, proving that science fact can be almost as strange as science fiction.

The writers also encouraged Homer’s intellectual side to flourish in “They Saved Lisa’s Brain,” an episode that has already been discussed in Chapter 7. After Stephen Hawking saves Lisa from a baying mob, the story ends with Professor Hawking chatting to Lisa’s father in Moe’s Tavern, where he is impressed with Homer’s ideas about cosmology: “Your theory of a doughnut-shaped universe is intriguing . . . I may have to steal it.”

This sounds ridiculous, but mathematically minded cosmologists claim that the universe might actually be structured like a doughnut. In order to explain how this geometry is possible, let us simplify the universe by imagining that the entire cosmos is flattened from three dimensions into two dimensions, so that everything exists on a sheet. Common sense might suggest that this universal sheet would be flat and extend to infinity in all directions. But cosmology is rarely a matter of common sense. Einstein taught us that space can bend, which leads to all sorts of other potential scenarios. For example, imagine that the universal sheet is not infinite, but instead has four edges, so that it looks rather like a large rectangular sheet of rubber. Next, imagine joining the two long edges of the sheet so it forms a cylinder, then connecting the two ends of the cylinder so that the whole sheet has been transformed into a hollow doughnut. This is exactly the sort of universe that Hawking and Homer were discussing.

If you lived on the surface of this doughnut universe, you could follow the grey arrow and eventually return to your original position.
The most obvious difference is that Homer’s statement concerns isosceles triangles, whereas the Pythagorean theorem relates to right triangles. You may remember from school that an isosceles triangle has two equal sides, whereas a right triangle has no restriction on the lengths of its sides, as long as one corner is a right angle.

There are two more problems in Homer’s statement. First, he talks about the “square roots” of lengths, whereas the Pythagorean theorem relies on the “squares” of lengths. Second, the Pythagorean theorem relates the hypotenuse (the longest side) of the right triangle to the other two sides, whereas Homer relates “any two sides” of the isosceles triangle to “the remaining side.” “Any two sides” could be the two equal sides or just one of the equal sides and the unequal side.

The diagrams and equations below summarize and highlight the differences between Homer’s statement and the Pythagorean theorem. Homer has taken a standard piece of mathematics and given it a twist, thereby creating a modification of the Pythagorean theorem, namely Simpson’s conjecture. The difference between a theorem and a conjecture is that the former has been proven to be true, whereas the latter is neither proven nor disproven . . . yet.

Simpson’s conjecture concerns all isosceles triangles, so if we try to prove it then we should need to show that it holds true for an infinity of triangles. However, if instead we try to disprove Simpson’s conjecture, then we would need to find just one triangle that defies the
ment about the lack of real knowledge in the population of viewers at large, implying that we are all ‘scarecrows’ as their little inside joke?”

Regardless of its origins and the motivations behind it, the Scarecrow conjecture is undoubtedly false, but it did inspire the trio of mathematicians at Augusta State to investigate the opposite of the Scarecrow conjecture, known as the crow conjecture, which states:

“The sum of the square roots of any two sides of an isosceles triangle is never equal to the square root of the remaining side.”

So, is Yick, Raﬁee, and Beasley’s crow conjecture true? We can test it by checking the two equations. Starting with equation (1), we can restate it and then rearrange it slightly:

\[ \sqrt{a} + \sqrt{a} \neq \sqrt{b} \]

\[ 2\sqrt{a} \neq \sqrt{b} \]

\[ 4a \neq b \]

\[ a \neq \frac{1}{2}b \]

This final equation states that it can never be true that the lengths \(a\) are only one-quarter of the base \(b\). Indeed, this must be the case, because \(a\) must be bigger than \(\frac{1}{2}b\), otherwise the three sides of the triangle will not touch each other. A quick look at the triangle above should make this obvious.
If \( n \) is the number of increments (i.e., the number of times per year that interest is calculated and added), then the following formula can be used to calculate the final sum \( (F) \) when the compound interest is also calculated at monthly, weekly, daily, and even hourly intervals:

\[
F = \$(1 + \frac{1}{n})^n
\]

<table>
<thead>
<tr>
<th>Initial sum</th>
<th>Annual interest</th>
<th>Time increment</th>
<th>Number of increments ((n))</th>
<th>Incremental interest</th>
<th>Final sum ((F))</th>
</tr>
</thead>
<tbody>
<tr>
<td>$1.00</td>
<td>100%</td>
<td>1 year</td>
<td>1</td>
<td>100.00%</td>
<td>$2.00</td>
</tr>
<tr>
<td>$1.00</td>
<td>100%</td>
<td>( \frac{1}{2} ) year</td>
<td>2</td>
<td>50.00%</td>
<td>$2.25</td>
</tr>
<tr>
<td>$1.00</td>
<td>100%</td>
<td>( \frac{1}{4} ) year</td>
<td>4</td>
<td>25.00%</td>
<td>$2.4414…</td>
</tr>
<tr>
<td>$1.00</td>
<td>100%</td>
<td>1 month</td>
<td>12</td>
<td>8.33%</td>
<td>$2.6130…</td>
</tr>
<tr>
<td>$1.00</td>
<td>100%</td>
<td>1 week</td>
<td>52</td>
<td>1.92%</td>
<td>$2.6925…</td>
</tr>
<tr>
<td>$1.00</td>
<td>100%</td>
<td>1 day</td>
<td>365</td>
<td>0.27%</td>
<td>$2.7145…</td>
</tr>
<tr>
<td>$1.00</td>
<td>100%</td>
<td>1 hour</td>
<td>8,760</td>
<td>0.01%</td>
<td>$2.7181…</td>
</tr>
</tbody>
</table>

By the time compound interest is calculated on a weekly basis, we are almost $0.70 better off than if we had been earning only simple annual interest. However, after this point, calculating the compound interest even more frequently achieves only one or two more pennies. This leads us to the fascinating question that began to obsess mathematicians: If the compound interest could be calculated not just every hour, not just every second, not just every microsecond, but at every moment, what would be the final sum at the end of the year?

The answer turns out to be 2.718281828459045235360287471352662497757247093699959574966967627724076630353547594571382178525166427… As you can probably guess, the decimal places continue to infinity, so this is an irrational number, and it is the number that we call \( e \).

2.718… was named \( e \) because it relates to exponential growth, which describes the surprising rate of growth experienced when money gathers interest year after year, or when anything repeatedly grows by a fixed rate again and again. For example, if the investment did increase
imagine the third dimension. The animated reality of Springfield is slightly more complicated than this, because we regularly see Homer and his family crossing behind and in front of each other, which ought to be impossible in a strictly two-dimensional universe. Nevertheless, for the purposes of this “Treehouse of Horror” segment, let us assume that Frink is correct in implying the existence of only two dimensions in *The Simpsons*, and let us see how he explains the concept of higher dimensions as he draws a diagram on the blackboard:

**Professor Frink:** Here is an ordinary square.

**Chief Wiggum:** Whoa, whoa! Slow down, egghead!

**Professor Frink:** But suppose we extend the square beyond the two dimensions of our universe along the hypothetical $z$-axis . . . There.

**Everyone:** [gasps]

**Professor Frink:** This forms a three-dimensional object known as a *cube*, or a *Frinkahedron* in honor of its discoverer.

Frink’s explanation illustrates the relationship between two and three dimensions. In fact, his approach can be used to explain the relationship between all dimensions.

If we start with zero dimensions, we have a zero-dimensional point. This point can be pulled in, say, the $x$ direction to trace a path that forms a one-dimensional line. Next, the one-dimensional line can be pulled in the perpendicular $y$ direction to form a two-dimensional square. This is where Professor Frink’s explanation picks up, because
A few moments later, a second mathematical tidbit appears in the three-dimensional landscape, courtesy of writer David S. Cohen:

$$1,782^{12} + 1,841^{12} = 1,922^{12}$$

This is yet another false solution to Fermat’s last theorem, just like the one created by Cohen for “The Wizard of Evergreen Terrace,” which was discussed in chapter 3. The numbers have been carefully chosen so that the two sides of the equation are almost equal. If we match the sum of the first two squares to the sum of the third square, then the results are accurate for the first nine digits, as shown in bold:

$$1,025,397,835,622,633,634,807,550,462,948,226,174,976 \quad (1,782^{12})$$

$$+ \quad 1,515,812,422,991,955,541,481,119,495,194,202,351,681 \quad (1,841^{12})$$

$$= \quad 2,541,210,259,314,801,410,819,278,649,643,651,567,616 \quad (1,922^{12})$$

This means that the discrepancy in the equation is just 0.00000003 percent, but that is more than enough to make it a false solution. Indeed, there is a quick way to spot that $1,782^{12} + 1,841^{12} = 1,922^{12}$ is a false solution, without having to do any lengthy calculations. The trick is to notice that we have an even number (1,782) raised to the twelfth power added to an odd number (1,841) raised to the twelfth power supposedly equaling an even number (1,922) raised to the twelfth power. The oddness and evenness are important because an odd number raised to any power will always give an odd result, whereas an even number raised to any power will always give an even result. Since an odd number added to an even number always gives an odd result, the left side of the equation is doomed to be odd, whereas the right side of the equation must be even. Therefore, it should be obvious that this is a false solution:

$$\text{even}^{12} + \text{odd}^{12} \neq \text{even}^{12}$$
a so-called *superposition of states*, which means it is both dead and alive . . . until the box is opened, at which point the situation is resolved.

Schrödinger and his cat make a guest appearance in another episode, which is titled “Law and Oracle” (2011). Traffic cops chase after a speeding Schrödinger, who eventually crashes. When he emerges from the wreckage, he is questioned about the box in his car. The cops are URL (pronounced Earl) and Fry, who has temporarily left his job at Planet Express.

**URL:** What’s in the box, Schrödinger?

**Schrödinger:** Um . . . A cat, some poison, und a cesium atom.

**Fry:** The cat! Is it alive or dead? Alive or dead?!

**URL:** Answer him, fool.

**Schrödinger:** It’s a superposition of both states until you open it and collapse the wave function.

**Fry:** Says you.

[Fry opens the box and a cat jumps out of it, attacking him. URL takes a close look at the box.]

**URL:** There’s also a lotta drugs in there.

Of course, this is a book about mathematics, not physics, so it is time to focus on the dozens of scenes in *Futurama* involving everything from convoluted geometry to incredible infinities. One such scene appears in “The Honking” (2000), which tells the story of Bender returning to his late uncle Vladimir’s haunted castle in order to attend the reading of Vladimir’s will. As the robot sits with his friends in the library, the digits 0101100101 appear on the wall, written in blood. Bender is more confused than spooked, but when he sees the digits reflected in the mirror—1010011010—he is immediately terrified.

Although no explanation is given in the dialogue, binary-savvy viewers would have appreciated the horrific significance of this scene.
the branch of applied mathematics that deals with code making and code breaking.

For example, several episodes contain billboards, notes, or graffiti that display messages written in alien scripts. The simplest alien script appears in “Lethal Inspection” (2010), when we see a note that reads:

Cryptographers call this a substitution cipher, because every letter of the English alphabet has been replaced by a different character, in this case an alien symbol. This type of cipher was first cracked by the ninth-century Arab mathematician Abu al-Kindi, who realized that every letter has a personality. Moreover, the personality of a particular letter is adopted by whichever symbol replaces that letter in the coded message. By spotting these traits, it is possible to decipher the message.

For example, frequency is an important part of a letter’s personality. e, t, and a are the three most frequent letters in English, while the most common symbols in the alien message are ⊱ and ⊴, which both appear six times. Hence, ⊱ and ⊴ probably represent e, t, or a, but which is which? A helpful clue appears in the first word, ⊱ ⊱ ⊱, which has a repeated ⊴. There are few words that fit the pattern *aa* or *tt*, but there are lots of words of the form *ee*, such as been, seen, teen, deer, feed, and fees. Hence, it is fair to assume that ⊴ = e. With a
Upon receiving Ramanujan’s papers, Hardy’s reaction veered between “fraud” and so brilliant that it was “scarcely possible to believe.” In the end, he concluded that the theorems “must be true, because, if they were not true, no one would have the imagination to invent them.” Hardy dubbed Ramanujan “a mathematician of the highest quality, a man of altogether exceptional originality and power,” and he began to make arrangements for the young Indian, still only twenty-six, to visit Cambridge. Hardy took great pride in being the man who had rescued such raw talent, and would later call it “the one romantic incident in my life.”

The two mathematicians finally met in April 1914, and their resulting partnership gave rise to discoveries in several areas of mathematics. For example, they made major contributions toward understanding a mathematical operation known as partition. As the name implies, partitioning concerns dividing up a number of objects into separate groups. The key question is, for a given number of objects, how many different ways can they be partitioned? The boxes below show that there is one way to partition one object, but there are five ways to partition four objects:

<table>
<thead>
<tr>
<th>Objects</th>
<th>Partitions</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>3</td>
<td>3</td>
</tr>
<tr>
<td>4</td>
<td>5</td>
</tr>
</tbody>
</table>

It is easy to find the number of partitions for a small quantity of objects, but it becomes trickier and trickier with more and more objects. This is because the number of possible partitions balloons in a rapid and erratic fashion. 10 objects can be partitioned in just 42 ways, but 100 objects can be partitioned in 190,569,292 ways. And
In “Möbius Dick”, the Planet Express ship is traveling through the galaxy and inadvertently enters the Bermuda Tetrahedron, a spaceship graveyard containing dozens of famous lost vessels. The Planet Express crew decide to investigate the region, whereupon they are attacked by a fearsome four-dimensional space whale, which Leela nicknames Möbius Dick.

The space whale’s name is both a play on Herman Melville’s novel *Moby-Dick* and a reference to a bizarre mathematical object known as a *Möbius strip* or *Möbius band*. The Möbius strip was discovered independently by the nineteenth-century German mathematicians August Möbius and Johann Listing. Using their simple recipe, you can build one for yourself. You will require:

(a) a strip of paper,

(b) sticky tape.

First, take the strip and twist one end through half a turn, as shown below. Then tape the two ends together to create the Möbius strip. That is all. A Möbius strip is essentially just a loop with a twist.
has a subplot that involves Bender turning himself into a home brewery. He gets the idea after he and his Planet Express colleagues visit a 711 convenience store to buy some alcohol. The store stocks Bender’s usual tipple, Olde Fortran malt liquor, named in honor of FORTRAN (FORmula TRANslation), a computer-programming language developed in the 1950s. The shelves are also stacked with St. Pauli’s Exclusion Principle Girl beer, which combines the name of an existing beer (St. Pauli Girl) with one of the foundations of quantum physics (the Pauli exclusion principle). Most interesting of all is a third brew called Klein’s, which comes in a strange flask. Aficionados of weird geometry will recognize that this is a **Klein bottle**, which is closely related to the Möbius strip.

The beer is called Klein’s in honor of Felix Klein, one of the greatest German mathematicians of the nineteenth century. His destiny may have been dictated the moment he was born, because each element of his date of birth, April 25, 1849, is the square of a prime number:

\[
\begin{align*}
\text{April} & \quad 25 \quad 1849 \\
4 & \quad 25 \quad 1,849 \\
2^2 & \quad 5^2 \quad 43^2
\end{align*}
\]
Klein’s research ranged across several areas, but he is most famous for the so-called Klein bottle. As with the Möbius strip, it will be easier to understand the shape and structure of a Klein bottle if you construct your own. You will require:

(a) a sheet of rubber,
(b) some sticky tape,
(c) a fourth dimension.

If, like me, you do not have access to a fourth dimension, then you can imagine how we might theoretically build a pseudo–Klein bottle in three dimensions. First, imagine rolling the rubber sheet into a cylinder and taping it along its length as shown on the next page in the first diagram. Then mark the two ends of the cylinder with arrows going in opposite directions. Next, and this is the tricky step, you must introduce a twist in the cylinder so that you can connect the two ends with both arrows heading in the same direction.

This is where the fourth dimension would come in very useful, but instead you will have to make do with a minor fudge. As shown in the middle two diagrams, bend the cylinder back on itself, and then imagine pushing one end of the cylinder through the wall of the self-same cylinder and up the inside. Finally, after this self-intersection step, roll the penetrating end of the cylinder downward, as in the fourth diagram, in order to connect the two ends of the cylinder. Crucially, when this connection is made, the arrows on each end of the cylinder will be pointing in the same direction.
of the bottle. As the arrow moves forward, it soon passes its starting position, except that now it is inside the bottle. If the arrow continues its journey up toward the neck and down again to the base, it then returns to the outer surface and eventually arrives back at its original position. Because the arrow is able to journey smoothly between the inner and outer surfaces of the Klein bottle, this indicates that the two surfaces are actually both part of the same surface.

Of course, without a well-defined inside and outside, the Klein bottle fails one of the main criteria required for a fully functioning bottle. After all, how can you put beer \textit{in} a Klein bottle, when \textit{in} is the same as \textit{out}?

In fact, Klein never called his creation a bottle. It was originally called a \textit{Kleinsche Fläche}, meaning a “Klein surface,” which is appropriate as it consists of a single surface. However, English-speaking mathematicians probably misheard this as \textit{Kleinsche Flasche}, which translates into English as “Klein bottle,” and the name stuck.

Finally, returning to a point raised earlier, the Klein bottle and the Möbius strip are closely related to each other. The most obvious connection is that both the strip and the bottle share the curious property of having only one surface. A second, and less obvious, connection is that a Klein bottle sliced into two halves creates a pair of Möbius strips.
Although there are only seven switches in total, the consequences of this mental juggling are very confusing. One way to keep track of what is happening is by drawing a Seeley diagram, invented by Dr. Alex Seeley, a Futurama fan living in London. A quick glance at this diagram reveals that the seven mind-switches eventually result in the Professor’s body containing Leela’s mind, Leela’s body containing Hermes’s mind, and so on.

As the episode draws to a close, everyone grows bored with the novelty and wants to return to his, her, or its original body. Alas, there is a major problem caused by a glitch in the Mind-switcher: Once two
considered a set because there is a mind for every body, but the minds and bodies are jumbled.

Having identified the sets, Keeler added two fresh people to the mix, Bubblegum Tate and Sweet Clyde, who then unmuddle the two sets one at a time. To see this in action, let us start with the smaller set and unmuddle it.

The Seeley diagram below tracks exactly what happens in the episode. We can see that the unmuddling phase begins with Sweet Clyde mind-switching with Fry (who has Zoidberg’s mind), then Bubblegum Tate mind-switches with Zoidberg (who has Fry’s mind). With two more mind-switches, Fry’s mind is returned to his own body and Zoidberg’s mind is returned to his own body.

Sweet Clyde and Bubblegum Tate are still mixed up, and the obvious next step would be to put their minds back in their correct bodies by performing one more mind-switch—this would be allowed, because they have not yet switched with each other. However, that would be a premature switch. The mathcketball geniuses were introduced as fresh people to unmuddle sets, and their work is not yet complete. So they must remain mixed up until they have dealt with the second set.
The Seeley diagram below tracks the nine mind-switches that occur as the second set is unmuddled. There is no need to go through the Seeley diagram switch by switch, but the overall pattern shows how the addition of Sweet Clyde and Bubblegum Tate creates the wiggle room required to resolve the situation. They are involved in every single mind-switch, which explains why the lowest quarter of the diagram looks so much busier than the region above it. Sweet Clyde and Bubblegum Tate act as temporary vessels for minds looking for the right home. As soon as they receive a mind, they switch it so that the mind ends up in the appropriate body. Whichever mind they then receive, they immediately pass it on to the appropriate body in the next switch, and so on.

Although Keeler did an excellent job of solving the mind-switching riddle and developing the Futurama theorem, it is important to point out that he either missed a trick, or ignored it in order to make
ARITHMETICKLE AND GEOMETEEHEEEHEE EXAMAMINATION

A FIVE-PART TEST OF HUMOR AND MATHEMATICS

The examination is divided into five separate sections.

The first section is an elementary examination, consisting of eight simple jokes.*

Subsequent sections are increasingly difficult.

Score yourself according to the number of laughs/groans you experience.

If you laugh/groan enough to score more than 50 percent, then you will have passed that particular section of the exam.

* These puns, gags, and shaggy-dog stories have been handed down from one generation of geeks to the next, which means that the names of the writers have sadly been lost in the mists of time (or the writers have understandably sought anonymity).
Joke 1  Q: What did the number 0 say to the number 8?  2 points
A: Nice belt!

Joke 2  Q: Why did 5 eat 6?  2 points
A: Because 7 8 9.

Joke 3  Knock, knock.  3 points
Who’s there?
Convex.
Convex who?
Convex go to prison!

Joke 4  Knock, knock.  3 points
Who’s there?
Prism.
Prism who?
Prism is where convex go!

Joke 5  Teacher: “What is seven Q plus three Q?”  2 points
Student: “Ten Q.”
Teacher: “You’re welcome.”
Joke 6  A Cherokee chief had three wives, each of whom was pregnant. The first squaw gave birth to a boy, and the chief was so elated that he built her a teepee made of buffalo hide. A few days later, the second squaw gave birth, and also had a boy. The chief was extremely happy; he built her a teepee made of antelope hide. The third squaw gave birth a few days later, but the chief kept the birth details a secret.

He built the third wife a teepee out of hippopotamus hide and challenged the people in the tribe to guess the details of the birth. Whoever in the tribe could guess correctly would receive a fine prize. Several people tried, but they were unsuccessful in their guesses. Finally, a young brave came forth and declared that the third wife had delivered twin boys. “Correct!” cried the chief. “But how did you know?”

“It’s simple,” replied the warrior. “The value of the squaw of the hippopotamus is equal to the sons of the squaws of the other two hides.”

Other versions of joke 6 have different punch lines. There are bonus points if either of these punch lines make you smile:

Joke 7  “The share of the hypertense muse equals the sum of the shares of the other two brides.” 2 points

Joke 8  “The squire of the high pot and noose is equal to the sum of the squires of the other two sides.” 2 points

T O T A L – 2 0 P O I N T S
Joke 1  Q: What are the 10 kinds of people in the world?  1 point
A: Those who understand binary, and those who don’t.

Joke 2  Q: Which trigonometric functions do farmers like?  1 point
A: Swine and cowswine.

Joke 3  Q: Prove that every horse has an infinite number of legs.  2 points
A: Proof by intimidation: Horses have an even number of legs. Behind they have two legs and in front they have forelegs. This makes a total of six legs, but this is an odd number of legs for a horse. The only number that is both odd and even is infinity. Therefore horses have an infinite number of legs.

Joke 4  Q: How did the mathematician reply when he was asked how his pet parrot died?  2 points
A: Polynomial. Polygon.
Joke 5  Q: What do you get when you cross an elephant and a banana?  3 points
A: $|\text{elephant}| \times |\text{banana}| \times \sin \theta$

Joke 6  Q: What do you get if you cross a mosquito with a mountain climber?  3 points
A: You can’t cross a vector with a scalar.

Joke 7  One day, Jesus said to his disciples: “The Kingdom of Heaven is like $2x^2 + 5x - 6$.”  2 points
Thomas looked confused and asked Peter: “What does the teacher mean?”
Peter replied: “Don’t worry— it’s just another one of his parabolas.”

Joke 8  Q: What is the volume of a pizza of thickness $a$ and radius $z$?  3 points
A: $\pi z^2 a$

Joke 9  During a security briefing at the White House, Defense Secretary Donald Rumsfeld breaks some tragic news: “Mr President, three Brazilian soldiers were killed yesterday while supporting U.S. troops.”
“My God!” shrieks President George W. Bush, and he buries his head in his hands. He remains stunned and silent for a full minute. Eventually, he looks up, takes a deep breath, and asks Rumsfeld: “How many is a brazillion?”

**TOTAL – 20 POINTS**
Examination III
University Senior Paper

Joke 1  Q: Why do computer scientists get Halloween and Christmas mixed up?  2 points
A: Because Oct. 31 = Dec. 25.

Joke 2  If the Teletubbies are a product of time and money, then:  4 points

\[
\text{Teletubbies} = \text{Time} \times \text{Money}
\]

\[
\text{But, Time} = \text{Money}
\]

\[\Rightarrow \text{Teletubbies} = \text{Money} \times \text{Money}\]

\[\Rightarrow \text{Teletubbies} = \text{Money}^2\]

Money is the root of all evil

\[\therefore \text{Money} = \sqrt{\text{Evil}}\]

\[\therefore \text{Money}^2 = \text{Evil}\]

\[\Rightarrow \text{Teletubbies} = \text{Evil}\]

Joke 3  Q: How hard is counting in binary?  2 points
A: It is as easy as 01 10 11.

Joke 4  Q: Why should you not mix alcohol and calculus?  2 points
A: Because you should not drink and derive.
Joke 5  
Student: “What's your favorite thing about mathematics?”  
Professor: “Knot theory.”  
Student: “Yeah, me neither.”

Joke 6  
When the Ark eventually lands after the Flood, Noah releases all the animals and makes a proclamation: “Go forth and multiply.”  

Several months later, Noah is delighted to see that all the creatures are breeding, except a pair of snakes, who remain childless. Noah asks: “What’s the problem?” The snakes have a simple request of Noah: “Please cut down some trees and let us live there.”  

Noah obliges, leaves them alone for a few weeks and then returns. Sure enough, there are lots of baby snakes. Noah asks why it was important to cut down the trees, and the snakes reply: “We're adders, and we need logs to multiply.”

Joke 7  
Q: If \( \lim_{x \to 8} \frac{1}{x - 8} = \infty \)  
then solve the following:  
\[ \lim_{x \to 5} \frac{1}{x - 5} = ? \]

A: \( \infty \)

TOTAL – 20 POINTS
Joke 1  Q: What’s a polar bear?  
A: A rectangular bear after a coordinate transformation.

Joke 2  Q: What goes “Pieces of seven! Pieces of seven!”?  
A: A parroty error.

Joke 3  Russell to Whitehead: “My Gödel is killing me!”

Joke 4  Q: What’s brown, furry, runs to the sea, and is equivalent to the axiom of choice?  
A: Zorn’s lemming.

Joke 5  Q: What’s yellow and equivalent to the axiom of choice?  
A: Zorn’s lemon.

Joke 6  Q: Why is it that the more accuracy you demand from an interpolation function, the more expensive it becomes to compute?  
A: That’s the law of spline demand.
Joke 7  Two mathematicians, Isaac and Gottfried, are in a pub. Isaac bemoans the lack of mathematical knowledge among the general public, but Gottfried is more optimistic. To prove his point, Gottfried waits until Isaac goes to the bathroom and calls over the barmaid. He explains that he is going to ask her a question when Isaac returns, and the barmaid simply has to reply: “One third x cubed.”

She replies: “Won thud ex-what?”

Gottfried repeats the statement, but more slowly this time: “One . . . third . . . x . . . cubed.”

The barmaid seems to get it, more or less, and walks away muttering over and over again: “Won thud ex-cubed.”

Isaac returns, he downs another drink with Gottfried, the argument continues and eventually Gottfried asks over the barmaid to prove his point: “Isaac, let’s try an experiment. Miss, do you mind if I ask you a simple calculus question? What is the integral of x?”

The barmaid stops, scratches her head, and hesitantly regurgitates: “Won . . . thud . . . ex-cubed.” Gottfried smiles smugly, but just before the barmaid walks away she stares at the two mathematicians and says: “. . . plus a constant!”

TOTAL – 20 POINTS
Joke 1  Q: What’s purple and commutes? 1 point
       A: An abelian grape.

Joke 2  Q: What’s lavender and commutes? 1 point
       A: An abelian semigrape.

Joke 3  Q: What’s nutritious and commutes? 1 point
       A: An abelian soup.

Joke 4  Q: What’s purple, commutes, and is worshipped
       by a limited number of people? 1 point
       A: A finitely venerated abelian grape.

Joke 5  Q: What’s purple, dangerous, and commutes? 1 point
       A: An abelian grape with a machine gun.

Joke 6  Q: What’s big, grey, and proves the uncountabil-
       ity of the decimal numbers? 2 points
       A: Cantor’s diagonal elephant.

Joke 7  Q: What’s the world’s longest song? 2 points
       A: “ℵ₀ Bottles of Beer on the Wall.”

Joke 8  Q: What does the “B.” in Benoit B. Mandelbrot
       stand for? 4 points
       A: Benoit B. Mandelbrot.
Joke 9  Q: What do you call a young eigensheep?  
A: A lamb, duh!  

Joke 10  One day, ye director of ye royal chain mail factory 
was asked to submit a sample in order to try to 
win a very large order for chain mail tunics and 
leggings. 

Though the tunic sample was accepted, he was 
told that the leggings were too long. He submitted 
a new sample, and this time the leggings were 
better, but too short. He submitted yet another 
sample, and this time the leggings were better 
still, but too long again. 

Ye director called ye mathematician and asked 
for her advice. He tailored another pair of chain 
mail leggings according to her instructions, and 
this time the samples were deemed to be perfect. 

Ye director asked ye mathematician how she 
calculated the measurements, and she replied: “I 
just used the wire-trousers hem test of uniform 
convergence.” 

Joke 11  An infinite number of mathematicians walk into a 
bar. The bartender says, “What can I get you?” 
The first mathematician says, “I’ll have one-half of 
a beer.” The second mathematician says, “I’ll have 
one-quarter of a beer.” The third mathematician 
says, “I’ll have one eighth of a beer.” The fourth 
mathematician says, “I’ll have one-sixteenth . . .” 
The bartender interrupts them, pours out a single 
beer and replies, “Know your limits.” 

T O T A L  -  2 0  P O I N T S
the branch of applied mathematics that deals with code making and code breaking.

For example, several episodes contain billboards, notes, or graffiti that display messages written in alien scripts. The simplest alien script appears in “Lethal Inspection” (2010), when we see a note that reads:

Cryptographers call this a substitution cipher, because every letter of the English alphabet has been replaced by a different character, in this case an alien symbol. This type of cipher was first cracked by the ninth-century Arab mathematician Abu al-Kindi, who realized that every letter has a personality. Moreover, the personality of a particular letter is adopted by whichever symbol replaces that letter in the coded message. By spotting these traits, it is possible to decipher the message.

For example, frequency is an important part of a letter’s personality. e, t, and a are the three most frequent letters in English, while the most common symbols in the alien message are $\varnothing$ and $\varpi$, which both appear six times. Hence, $\varnothing$ and $\varpi$ probably represent e, t, or a, but which is which? A helpful clue appears in the first word, $\varnothing \varpi \varpi \varnothing$, which has a repeated $\varpi$. There are few words that fit the pattern *aa* or *tt*, but there are lots of words of the form *ee*, such as been, seen, teen, deer, feed, and fees. Hence, it is fair to assume that $\varpi = e$. With a
bit more detective work, it would be possible to unravel this particular message: *Need extra cash? Melt down your old unwanted humans. We pay top dollar.* And with one or two more messages the entire alien script could be deciphered from A (Ϝ) to Z (Ϛ).

Not surprisingly, mathematically adept *Futurama* fans found this alien code trivial to crack, so Jeff Westbrook (who has written for both *Futurama* and *The Simpsons*) developed a more complex alien code.

Westbrook’s efforts resulted in reinventing the text autokey cipher, which is akin to a cipher first devised by Girolamo Cardano (1501–76), one of the greatest Italian Renaissance mathematicians. The cipher operates by first assigning numbers to the letters of the alphabet: A = 0, B = 1, C = 2, D = 3, E = 4, …, Z = 25. After this preliminary step, encryption requires just two more steps. First, each letter is replaced with the numerical total of all the letters in all the words up to and including the letter itself. Hence, BENDER OK is transformed as follows:

<table>
<thead>
<tr>
<th>Letter</th>
<th>B</th>
<th>E</th>
<th>N</th>
<th>D</th>
<th>E</th>
<th>R</th>
<th>O</th>
<th>K</th>
</tr>
</thead>
<tbody>
<tr>
<td>Number</td>
<td>1</td>
<td>4</td>
<td>13</td>
<td>3</td>
<td>4</td>
<td>17</td>
<td>14</td>
<td>10</td>
</tr>
<tr>
<td>Total</td>
<td>1</td>
<td>5</td>
<td>18</td>
<td>21</td>
<td>25</td>
<td>42</td>
<td>56</td>
<td>66</td>
</tr>
</tbody>
</table>

The second and final encryption step involves replacing each total number with the corresponding symbol from this list:

There are only 26 symbols, which are associated with the numbers 0 to 25, so what symbol represents R, O, and K, which have just been
assigned totals of 42, 56, and 66, respectively? The rule* is that numbers bigger than 25 are reduced by 26 again and again until they are in the range 0 to 25. Hence, to find the symbol for R, we subtract 26 from 42, which leaves us with 16, which is associated with \( \heartsuit \). By applying the same rule to the remaining two letters, BENDER OK is encrypted as \( \heartsuit \heartsuit \) \( \heartsuit \). However, if it was preceded by some other words, then BENDER OK would be encrypted differently, as the running total would be affected. This made Westbrook’s autokey cipher fiendishly difficult to crack. He used it to encode various messages across several episodes, and they proved to be a serious challenge to those Futurama fans who made a hobby out of cracking the codes that appeared in the series. Indeed, it took a year before anybody cracked the exact details of the autokey cipher and decoded the various messages.

Although one might expect some challenging codes to appear in the Futurama episode “The Duh-Vinci Code” (2010), its most interesting mathematical aspect relates to a completely different area of mathematics. The plot involves the Planet Express team analyzing the fine detail of Leonardo da Vinci’s painting The Last Supper, whereupon they notice something odd about James the Lesser, one of the apostles sitting at the left end of the table. A high-powered X-ray reveals that da Vinci originally painted James as a wooden robot. In order to find out whether or not James was an early automaton, the crew heads to Future-Roma, where they discover St. James’s tomb. Importantly, they also stumble upon a crypt with an appropriately cryptic engraving that reads:

\[ \text{II}^{\text{XI}} - (\text{XXIII} \cdot \text{LXXXIX}) \]

* This rule belongs to a branch of mathematics known as modular arithmetic. As well as being very useful in the context of cryptography, modular arithmetic also plays a vital role in several other areas of mathematical research, including the proof of Fermat’s last theorem.